

IMAGES OF THE POLAR MAPS FOR HYPERSURFACES

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ABSTRACT. For a projective hypersurface $X \subset \mathbb{P}^n$, the images of the polar maps of degree k are studied. The cohomology class defined by these maps is calculated and classical results on dual varieties are presented as applications.

1. INTRODUCTION

For any smooth variety $X \subset \mathbb{P}^n$ of codimension q the classical gauss map is the map

$$g^1 : X \rightarrow \mathbb{G}^q(\mathbb{P}^n)$$

which associates to each point ξ the projective linear subspace of codimension q tangent to X at ξ in \mathbb{P}^n :

$$\xi \mapsto \overline{T_\xi X}$$

If X is a hypersurface defined by the set of zeros of a homogeneous polynomial F of degree d , $X = V(F) := \{x \in \mathbb{P}^n \mid F(x) = 0\} \subset \mathbb{P}^n$, then the gauss map has the following coordinate expression

$$(1) \quad \xi \mapsto V\left(\sum \frac{\partial F}{\partial x_i}(\xi) x_i\right)$$

If X has singularities we no longer have a map which is regular, but only a rational map.

We may think of the Gauss map in the following way: For a fixed point $p \in X$ we associate an algebraic cycle in \mathbb{P}^n of degree one which is obtained via the first partial derivatives of the polynomial defining X . If we take higher derivatives of this polynomial, then it is possible to associate to every point $p \in X$ an algebraic cycle of higher degree. For a fixed point p , this cycle has appeared in different contexts, we will mention two of them

1.1. Polar Transformations and Homaloidal Polynomials. The map (1) has a natural generalization taking higher derivatives. The following definition (c.f. [2], [3]) suggests such a generalization.

Definition 1.1. Let $\mathbf{p} = (p_0, \dots, p_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, and let $p = [p_0 : \dots : p_n]$ be the corresponding point in \mathbb{P}^n . For every positive integer $k < d$ the k -th polar

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polynomial of F is given by

$$(2) \quad \Delta_{\mathbf{p}}^s(F)(x) := \left(p_0 \frac{\partial}{\partial x_0} + \cdots + p_n \frac{\partial}{\partial x_n} \right)^{(s)} F(x)$$

where (s) denotes the symbolic power taking derivatives and products.

With this notation, the *reciprocity theorem* proved in [2] implies that the gauss map is given by

$$\xi \mapsto V \left(\Delta_{\xi}^{(d-1)}(F)(x) \right).$$

Furthermore, the image of the (rational) Gauss map is the *dual* (or polar) variety of X , an object which has been thoroughly studied in classical algebraic geometry.

1.2. Higher Fundamental Forms. In [1] (c.f. also [5] and [6]), Beheshti defined for a smooth point $p \in X$ the hypersurfaces Y_p^k as the zero set of the polynomial

$$(3) \quad \sum_{0 \leq i_1, \dots, i_k \leq n} \frac{\partial^k P}{\partial x_{i_1} \cdots \partial x_{i_k}}(p) x_{i_1} \cdots x_{i_k}$$

The hypersurface Y_p^1 is the tangent plane at p and the restriction of the hypersurface Y_p^2 to T_p is a quadratic form: the second fundamental form.

In this paper we will study the properties of the map

$$g^k : X \rightarrow \mathbb{C}_k^1(\mathbb{P}^n)$$

which associates to each point ξ the effective algebraic cycle of degree k and codimension 1 which approximates X at ξ .

As in the case of the Gauss map, the higher degree polar maps are only rational in general. Interestingly however, they can still be regular in the presence of certain singularities.

2. BASIC PROPERTIES

Let us recall the Euler relation:

$$(4) \quad d \cdot F = \sum_{i=0}^n \frac{\partial F}{\partial x_i} x_i$$

Iterating the Euler relation we get the following:

$$(5) \quad (d-1) \cdot \frac{\partial F}{\partial x_i} = \sum_{k=0}^n \frac{\partial^2 F}{\partial x_k \partial x_i} x_k$$

If we substitute equation (5) into equation (4) we get the following relation for F :

$$(6) \quad d \cdot (d-1) \cdot F = \sum_{i=0}^n \sum_{k=0}^n \frac{\partial^2 F}{\partial x_k \partial x_i} x_k x_i$$

In general, if $s \leq d = \deg(F)$ we have the following equation:

$$(7) \quad d(d-1) \cdots (d-s+1)F = \sum_{|\alpha|=s} \frac{\partial^{|\alpha|} F}{\partial x^\alpha} x^\alpha$$

where α runs over all multi-indices of length s , i.e.

$$\alpha = (\alpha_0, \dots, \alpha_n) \text{ with } \alpha_i \in \mathbb{N}$$

and

$$|\alpha| = \alpha_0 + \cdots + \alpha_n$$

$$\frac{\partial^{|\alpha|} F}{\partial x^\alpha} = \frac{\partial^{|\alpha|} F}{\partial x_{\alpha_0} \partial x_{\alpha_1} \cdots \partial x_{\alpha_n}}$$

Remark 2.1. One of the consequences of the Euler formula is that the systems $\{\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}, F\}$ and $\{\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}\}$ have the same set of solutions, more precisely, they define the same scheme since the ideals they generate are equal. Recursively we obtain the following lemma.

Lemma 2.2. *Let ξ be a point in \mathbb{P}^n such that $\frac{\partial^s F}{\partial x^\alpha}(\xi) = 0$ for all α with $|\alpha| = s$. Then $F(\xi) = 0$ and $\frac{\partial^{|\beta|} F}{\partial x^\beta}(\xi) = 0$ for all β with $|\beta| \leq s$*

Now we define the higher degree polar maps and we derive some consequences from the generalized Euler formulas given above.

Definition 2.3. For every $k \leq d$, the *degree k polar map*

$$(8) \quad g^k : X \dashrightarrow \mathbb{C}_k^1(\mathbb{P}^n)$$

is the rational map defined by

$$\xi \mapsto V \left(\sum_{|\alpha|=k} \frac{\partial^k F}{\partial x^\alpha}(\xi) x^\alpha \right)$$

The space $\mathbb{C}_k^1(\mathbb{P}^n)$ of cycles of codimension 1 and degree k in \mathbb{P}^n can be identified with $\mathbb{P}^{\binom{n+k}{k}-1}$, this identification is via the Chow coordinates. Every codimension 1 cycle is determined by a multivariable homogeneous polynomial of degree k . If a cycle is defined by a polynomial $\sum a_\alpha x^\alpha$ then its Chow coordinates are $[a_0 : \cdots : a_\alpha : \cdots]$. Using the Chow coordinates in $\mathbb{C}_k^1(\mathbb{P}^n) \cong \mathbb{P}^{\binom{n+k}{k}-1}$ the degree k Gauss map is just

$$\xi \mapsto \left[\frac{\partial^k F}{\partial x_0 \cdots \partial x_0}(\xi) : \cdots : \frac{\partial^k F}{\partial x^\alpha}(\xi) : \cdots : \frac{\partial^k F}{\partial x_n \cdots \partial x_n} \right]$$

As it was remarked in the introduction, the first polar map g^1 coincides with the classical projective gauss map.

The higher degree polar maps are only rational in general. Interestingly however, they can still be regular in the presence of certain singularities. More precisely, the following is true:

Theorem 2.4. *If a hypersurface of degree d has a regular polar map of degree p , it also has regular polar maps of degree q for $p \leq q \leq d$*

Proof. This follows immediately from the Euler relation: If the degree p polar map is regular, then for every $\xi \in X$ some p -th partial derivative $\frac{\partial^p F}{\partial x_\alpha}(\xi)$ is not zero. On the other hand, if all the q -th partial derivatives are zero at ξ then lemma (2.2) implies that $\frac{\partial^p F}{\partial x_\alpha}(\xi) = 0$ for all α ! \square

Example 2.5. Let $V \subset \mathbb{P}^2$ be the nodal plane cubic defined by $F(x_0, x_1, x_2) = x_2x_1^2 - x_0^3 - x_0^2x_2$, then V does not have a regular polar map of degree 1, but it has a well defined polar map of degree 2 (see figure 1):

$$\bar{\xi} \mapsto V_{\bar{\xi}} = V(-(3\xi_0 + \xi_2)x_0^2 - 2\xi_0x_0x_2 + \xi_2x_1^2 + 2\xi_1x_1x_2)$$

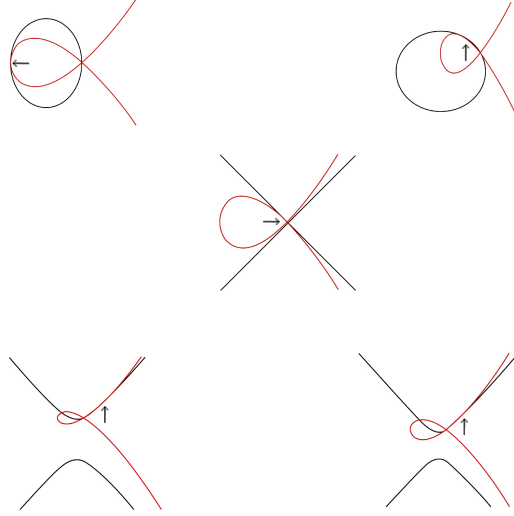


FIGURE 1. Second degree approximations to the nodal cubic: The red curve is the nodal cubic, the black curves are the conics approximating the curve at the point signaled by the arrow.

That is, to every point v we associate a quadric which approximates the curve at v . Notice that at the node $[0 : 0 : 1]$ we do have a well defined second order approximation: the union of the two possible tangents.

3. DEGREE AND DIMENSION

The closure of the image of the first polar map defines the dual variety of the hypersurface X . We could ask what are the general properties of the images of these higher degree polar maps. The following results extend some classic results of projective geometry.

Theorem 3.1. *If X is not a cone, then the $(d-1)$ polar map $g^{(d-1)}$ is an isomorphism from X into its image. (This extends the well known result for the duals of smooth quadrics. Note that any singular quadric is a cone.)*

In order to prove this theorem we will first prove a characterization of cones. Recall that a cone is the linear join $Y \# p$ of a variety $Y \in \mathbb{P}^n$ with a point $p \in \mathbb{P}^n$ such that $p \notin Y$.

Lemma 3.2. *$X \subset \mathbb{P}^n$ is a cone if and only if there is some $\xi \in X$ such that $\text{mult}_\xi X = d$ (recall that X is a hypersurface of degree d).*

Proof. If X is a cone, $X = Y \# p$ then p is a point such that $\text{mult}_p X = d$. Conversely, if ξ is a point of multiplicity d , let l be any line passing through ξ . If l intersects X at any other point q then $\overline{\xi q} = l \subset X$ (otherwise the degree of X would be greater than d). \square

Now we can prove theorem 3.1.

Proof. Notice that using Chow coordinates, $g^{(d-1)}$ is a linear map. That is, it is the map induced by a linear map $\tilde{L} : \mathbb{P}^n \rightarrow \mathbb{P}^{\binom{n+k}{k}-1}$ by restriction to X . This means that \tilde{L} itself is induced by a linear map $L : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{\binom{n+k}{k}}$. If $g^{(d-1)}$ is not an injection then L is certainly not an injection. We will show that this leads to a contradiction.

If L is not an injection then it has a non-zero kernel. Let $\bar{\xi}$ be a non-zero vector in that kernel. This means that all the $(d-1)$ -partial derivatives of F vanish at $\bar{\xi}$. But the generalized Euler relation implies then that $\bar{\xi} \in X$. This is a contradiction because we get a point $\xi \in X$ such that $\text{mult}_\xi X = d$ (because all the $(d-1)$ -partial derivatives vanish at ξ), i.e. X is a cone (by the previous lemma). \square

Lemma 3.3. *If the degree p polar Map is regular, then*

$$(g^p)^*(O(1)) = O_X(d-p)$$

Proof. First of all, notice that if g^p is regular, then it extends to a regular map defined on all of \mathbb{P}^n :

$$\tilde{g}^p : \mathbb{P}^n \rightarrow \mathbb{P}^N$$

We see this as follows. Note that the extension is given by the same coordinate functions. The reason for having this regular extension is the following: If there is some point $\xi \in \mathbb{P}^n$ where all the coordinate functions vanish simultaneously lemma (2.2) then implies that $F(\xi) = 0$, i.e. $\xi \in X$, but this would imply that g^p is not regular!

Now, since the coordinate functions of \tilde{g}^p can be interpreted as sections of the bundle $(\tilde{g}^p)^*(O(1))$ and they are polynomials of degree $d-p$, we get that the pullback of $O(1)$ under this map is $O(d-p)$. Since g^p is just the restriction of \tilde{g}^p we get the lemma. \square

Theorem 3.4. *If the degree p polar map is regular, the image of the p -th Gauss image variety is $n-1$ and the degree of the p -th Gauss image variety is $d(d-p)^{n-1}$. (This extends the classical formulas for the degree of the dual of a smooth hypersurface and the fact that the dual of a smooth hypersurface is a hypersurface).*

Proof. Using the previous lemma, this becomes just a chern class calculation. We will denote with $H_{\mathbb{P}^s}$ the class of a hyperplane in $H^2(\mathbb{P}^s, \mathbb{Z})$ and with $H_{\mathbb{P}^s}^k$ its k -fold cup product. Recall that $\dim X = n-1$. Using the previous lemma we obtain:

$$(g^p)^*(H_{\mathbb{P}^N}) = (d-p)H_{\mathbb{P}^n}$$

therefore

$$(g^p)^*(H_{\mathbb{P}^N}^{n-1}) = (d-p)^{(n-1)}H_{\mathbb{P}^n}^{(n-1)}$$

If \langle, \rangle denotes the Kronecker pairing, then the following calculation proves the result:

$$\begin{aligned} \deg g^p(X) &= \langle g^p(X), H_{\mathbb{P}^N}^{(n-1)} \rangle \\ &= \langle (\tilde{g}^p)_*(X), H_{\mathbb{P}^N}^{(n-1)} \rangle \\ &= \langle X, \tilde{g}^*(H_{\mathbb{P}^N}^{(n-1)}) \rangle \\ &= \langle X, (d-p)^{(n-1)}H_{\mathbb{P}^n}^{(n-1)} \rangle \\ &= d(d-p)^{(n-1)} \end{aligned}$$

□

Lemma (3.3) also allows us to prove the following calculation.

Theorem 3.5. *If X is a smooth hypersurface, the cohomology class defined by the p -th polar map $[\mathbf{g}^p] \in H^2(X)$ satisfies*

$$[\mathbf{g}^p] = \frac{d-p}{d-1}[\mathbf{g}^1] = \frac{d-p}{d-1}c(N_X(-1))$$

More generally,

$$[\mathbf{g}^p] = c_1(O_X(d-p))$$

Proof. The fundamental result of Lawson and Michelsohn in [7] implies that the homotopy class $[\mathbf{g}^p]$ coincides with $c_1((\mathbf{g}^p)^*(O(1)))$. Lemma (3.3) provides this last calculation:

$$c_1((\mathbf{g}^p)^*(O(1))) = c_1(O_X(d-p))$$

The first claim is a consequence of the adjunction formula:

$$O_X(d) = [X]|_X = N_X$$

But we can also write

$$O_X(d) = O_X(d-1) \otimes O_X(1) = [g^1] \otimes O_X(1)$$

therefore

$$[g^1] = N_X(-1)$$

□

4. EXAMPLES AND APPLICATIONS

The higher degree polar maps encode information about the underlying variety. For example if the p -th polar map is regular then the variety cannot have singularities of order p . The next theorem recovers the classical calculation of the number of flexes of a smooth plane curve. Recall that a flex of a plain curve C is a point $p \in C$ where the tangent line has contact of order greater than 2, that is, the local intersection number at p of the tangent line at p and the curve is greater than 2.

Theorem 4.1. *Let C be a smooth plane curve $C \subset \mathbb{P}^2$ of degree $d \geq 2$. Then C has $3d(d-2)$ flexes.*

Proof. Let C be defined by a homogeneous polynomial F . A point p is a flex if and only if the determinant of the Hessian matrix H_p is zero, where

$$H_p = \begin{pmatrix} \frac{\partial^2 F}{\partial x_0^2}(p) & \frac{\partial^2 F}{\partial x_0 x_1}(p) & \frac{\partial^2 F}{\partial x_0 x_2}(p) \\ \frac{\partial^2 F}{\partial x_0 x_1}(p) & \frac{\partial^2 F}{\partial x_1^2}(p) & \frac{\partial^2 F}{\partial x_1 x_2}(p) \\ \frac{\partial^2 F}{\partial x_0 x_2}(p) & \frac{\partial^2 F}{\partial x_1 x_2}(p) & \frac{\partial^2 F}{\partial x_2^2}(p) \end{pmatrix}$$

but H_p is the quadratic form which defines the second polar map at p , i.e.

$$(9) \quad g^2(p) = \xi^T H_p \xi$$

Notice that since C is smooth, the second polar map is regular. So the condition of p being a flex is exactly the same as $g^2(p)$ being a singular quadric. Now, singular quadrics form a hypersurface Δ of degree 3 in the space \mathbb{P}^5 of all degree 2 homogeneous polynomials in three variables. This hypersurface Δ is given by the vanishing of the determinant of the matrix defining a quadratic form.

Thus we are interested in computing the number of intersection points of $g^2(C)$ with Δ . But $\deg(g^2(C)) = d(d-2)$ and $\deg(\Delta) = 3$, therefore Bezout's theorem implies that the number of flexes is $3d(d-2)$. \square

The next example shows how it is possible to have a hypersurface X with degenerate gauss map (the image of the gauss map will be a curve) and nevertheless the second polar map has the same dimension as the hypersurface.

Remark 4.2. It would be interesting to find examples of hypersurfaces X with degenerate higher polar images such that X is not a cone. If we define a hypersurface to be k -defective if its k -th polar map has a lower dimension than X itself, then we may ask for a characterization of k -defective hypersurfaces (1-defective hypersurfaces coincide with the notion of defective hypersurfaces, i.e. having a dual which is not a hypersurface).

Example 4.3. Let $\Sigma \in \mathbb{P}^3$ be the rational normal curve. That is, Σ is the image of the rational parametrization

$$t \mapsto [1 : t : t^2 : t^3]$$

It is known that the dual variety Σ^\vee is a hypersurface defined by the discriminant of the general single variable polynomial of degree 3 (cf [4] ch. 1.), namely, the equation defining the dual hypersurface is:

$$\Delta = x_1^2 x_2^2 - 4x_1^3 x_3^2 - 27x_0^2 x_3^2 + 18x_0 x_1 x_2 x_3$$

Now, Σ^\vee must necessarily be singular, since otherwise the dual variety would be a hypersurface. But the singularities of Σ^\vee actually have order 1, therefore the second polar map is regular and using our calculations we can conclude that $g^2(\Sigma^\vee)$ is a surface of degree $4(4-2)^2 = 16$ in \mathbb{P}^9 .

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